CS 331, Fall 2025

Lecture 14 (10/15)

Today:
- Cuts
- Maxflow-mincut theorem
- Flow algos

Cuts (Part V, Section 4.1)

"Dual" problem to flows.

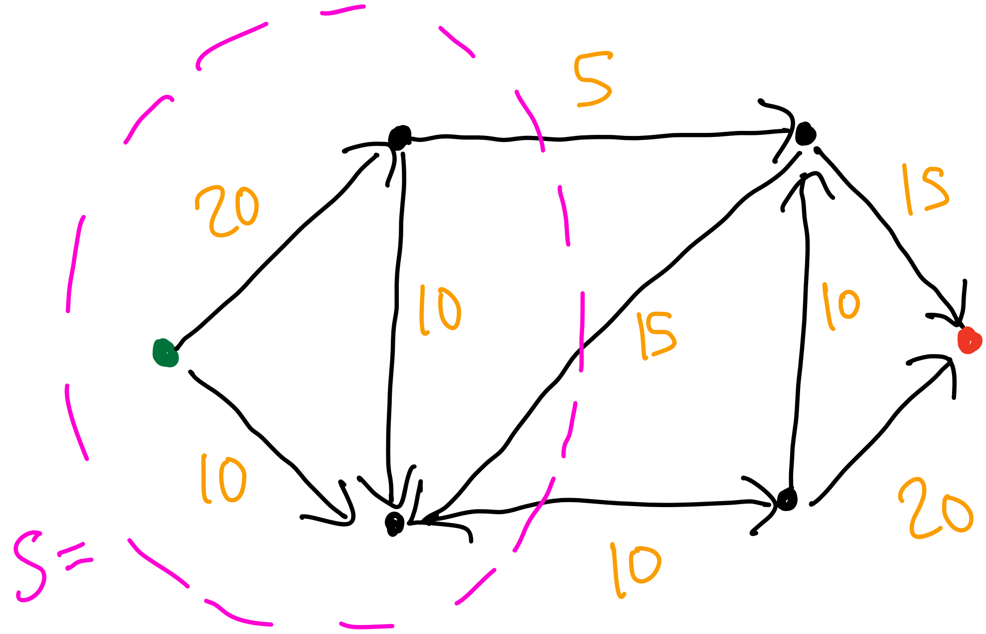
(Much more on this next week...)

For $S \subseteq V$:

$$\text{cut}(S) = \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} c_{(u,v)}$$

"total capacity crossing from S to $V \setminus S$"

Example



$cut(S) = 10 + 5 = 15$

We only track from $S \to V \setminus S$

$cut(V \setminus S) = 15$

Say that S is an s-t cut if

$$\underbrace{s \in S, \quad t \notin S}_{S \text{ separates } s \text{ from } t}$$

Historical context

Flows/cuts first formulated in classified report (1954) modeling Soviet network

Harris-Ross were interested in $\overset{\text{"cut"}}{\underline{\text{disrupting}}}$...

(Declassified by Schrijver, '05)

Maxflow-mincut theorems (Part V, Section 4.2)

max $\partial f(s)$

$0 \le f_e \le c_e$ (feasible)

$\partial f(v) = 0$ $\forall v \notin \{s, t\}$ (s-t flow)

s-t maxflow

min $\text{cut}(S)$

$S \subseteq V$

$s \in S$

$t \notin S$ (s-t cut)

s-t mincut

Claim: for <u>any</u> $G = (V, E, \underbrace{c}_{\text{positive}})$, $s \neq t \in V$

$$s\text{-}t \text{ maxflow} = s\text{-}t \text{ mincut}$$

(strong maxflow-mincut theorem)

Sanity Check: if no s-t path, both $= 0$

Weak maxflow-mincut theorem:

$$s\text{-}t \text{ maxflow} \leq s\text{-}t \text{ mincut}$$

Proof: Flow must get from S to $V \setminus S$

contains s contains t

It only has $\mathrm{cut}(S)$ to do so.

More formally, $\partial f(S) = \sum_{u \in S} \partial f(u)$

$$= \sum_{u \in S} \sum_{(u,v) \in E} f_{(u,v)} - \sum_{u \in S} \sum_{(v,u) \in E} f_{(v,u)}$$

$$= \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} f_{(u,v)} - \sum_{\substack{(v,u) \in E \\ u \in S \\ v \notin S}} f_{(v,u)}$$

$$\leq \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} c_{(u,v)} = \mathrm{cut}(S)$$

For strong maxflow-mincut:

need flow f, cut S s.t.

1) $f_{(u,v)} = c_{(u,v)} \quad \forall (u,v) \in E \quad \begin{matrix} u \in S \\ v \notin S \end{matrix}$

2) $f_{(v,u)} = 0 \quad \forall (v,u) \in E \quad \begin{matrix} u \in S \\ v \notin S \end{matrix}$

$\boxed{\text{Residual graph}}$ "what are all flows I can add s.t. stay feasible?"

Let $(u,v) \in E$.

- If $0 < f_{(u,v)} < c_{(u,v)}$:

Add (u,v) to G^f, capacity $= c_{(u,v)} - f_{(u,v)}$

(v,u) to G^f, capacity $= f_{(u,v)}$

- If $f_{(u,v)} = c_{(u,v)}$:

Only add backward edge

(v,u) to G^f, capacity $= f_{(u,v)}$

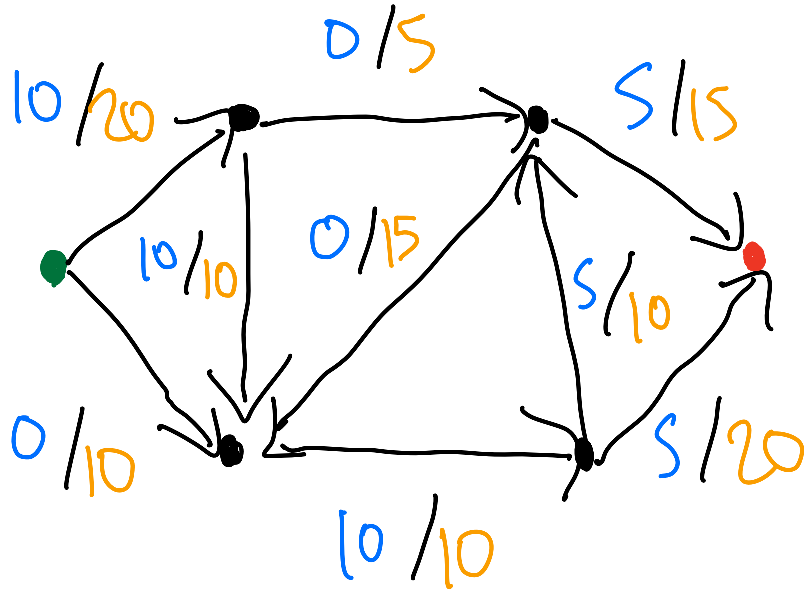
- If $f_{(u,v)} = 0$:

Only add forward edge

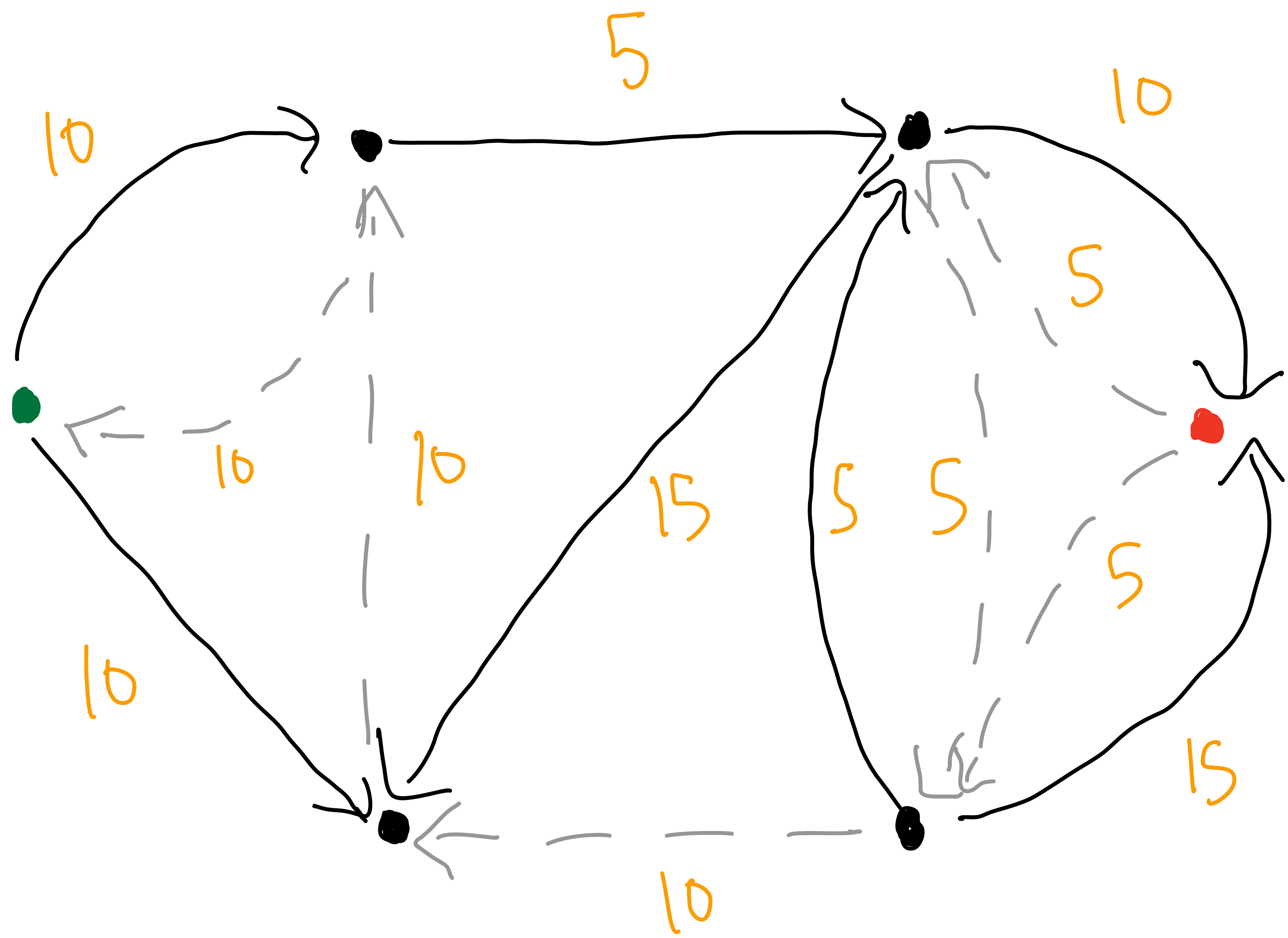
(u,v) to G, capacity $= f_{(u,v)}$

Examples

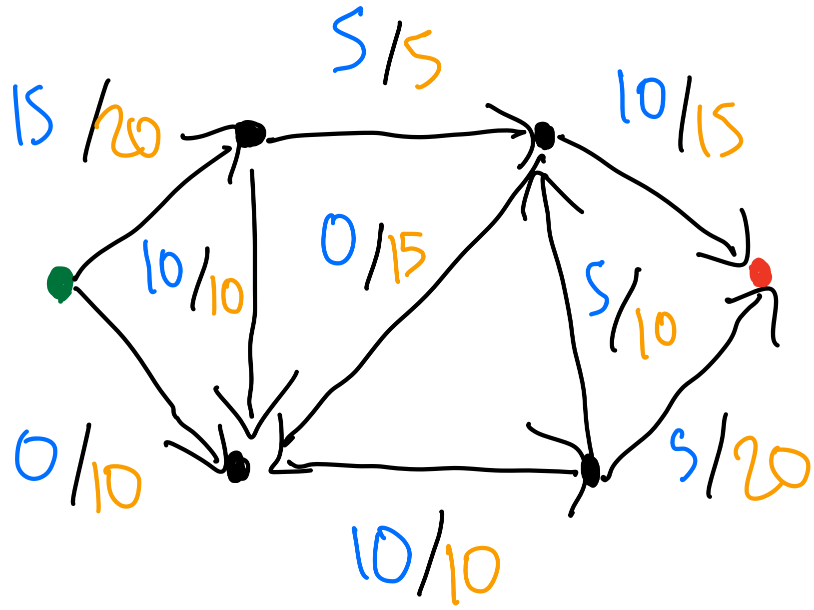
Flow on G:



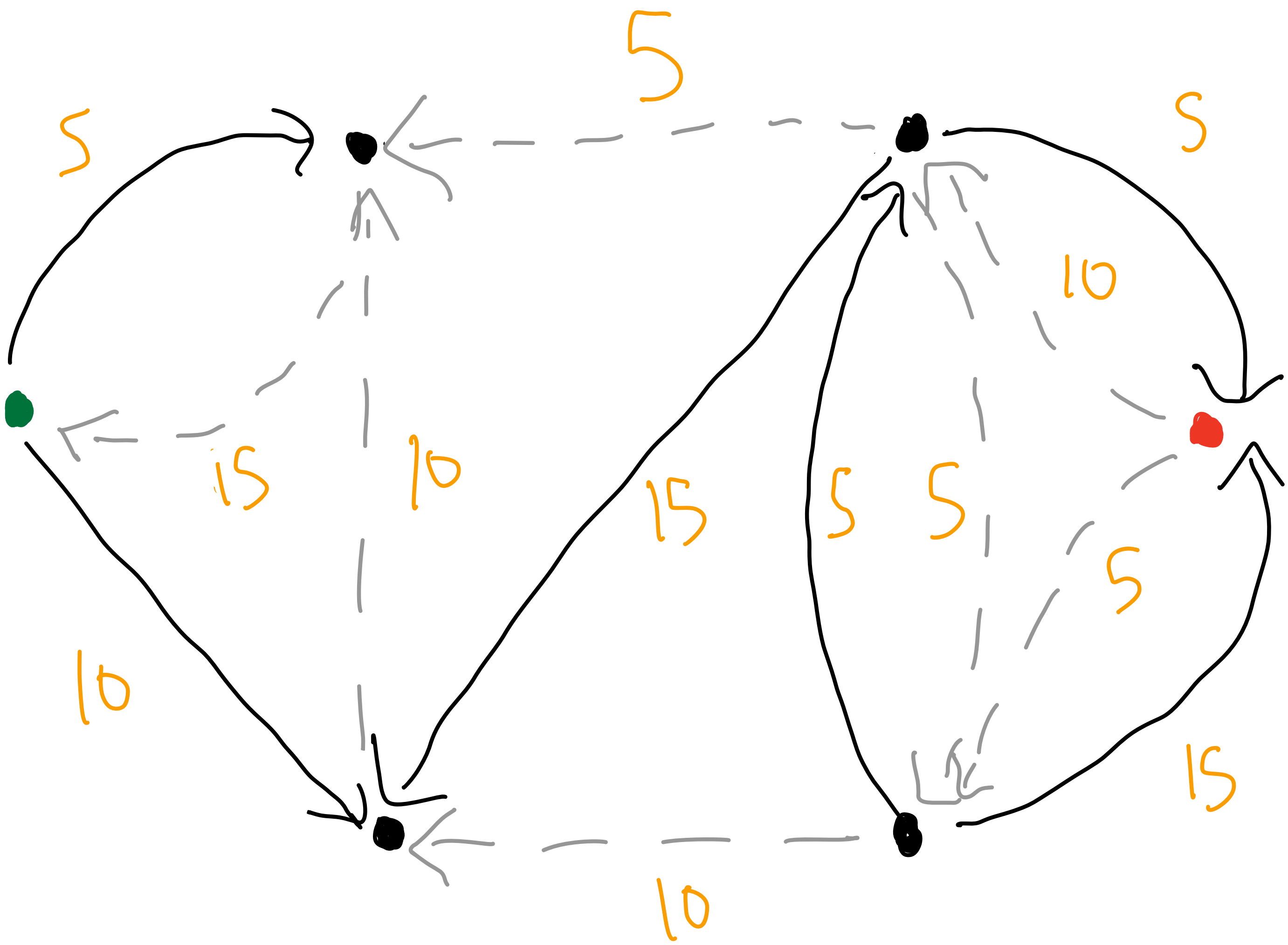
G^f:



Flow on G:



G^f:



Interestingly, s cannot reach t.

Proof of strong maxflow-mincut:

Plan: 1) If s-t path in G^f,
not maxflow

2) If no s-t path in G^f,
$\partial f(S) = cut(S)$
where S reachable from s

Proof 1): Every edge in G^f has positive capacity.

Let P be s-t path,

$w = width(P) > 0$

We can create a bigger flow.

$$f'_e = \begin{cases} f_e + w & e \in P \quad \text{(push } w \text{ flow along } P\text{)} \\ f_e & e \notin P \end{cases}$$

Still feasible (by construction).

Still s-t flow:

$$\partial f'(v) = \partial f(v) \underbrace{- w + w}_{\text{if } v \text{ participates in } P} = 0$$

$v \notin \{s, t\}$

$$\partial f'(s) = \partial f(s) + w$$
$$> \partial f(s)$$

Proof 2): Suppose s can't reach t.

Let S = reachable from s

$t \notin S$, no edges from $S \to V \setminus S$

Claim: f, S satisfy $\star$!!!

Let $u \in S$, $v \notin S$.

- If $(u,v) \in E$, $f_{(u,v)} = c_{(u,v)}$, or forward edge in G^f $\Rightarrow\Leftarrow$
- If $(v,u) \in E$, similarly $f_{(u,v)} = 0$

Hence $\partial f(s) = \mathrm{cut}(S)$ $\blacksquare$

Flow algos (Part V, Section 4.3)

Generic algos so far:

- Search: explore unexplored vertex
- Shortest path: relax tense edge
- Flow: push w units along augmenting path

s-t path $\in G^f$

MaxFlow(G, s, t):

$f \leftarrow$ all-zeroes vector in $\mathbb{R}^E$

While $\exists P$, s-t path in G^f:

  $w \leftarrow \text{width}(P)$

  $f \leftarrow f + (w \text{ along } P)$

Return f

What path?

$\boxed{\text{Idea 1: any path}}$

Suppose all capacities integers.

Invariant: all capacities in G^f integers.

Thus can always push $w \geq 1$ flow/iter.

Runtime analysis:

Let $F^* =$ maxflow value.

$$\underbrace{F^*}_{\#\text{ iters}} \times \underbrace{O(m)}_{\substack{\text{cost of finding} \\ \text{path, e.g. BFS}}} = O(mF^*)$$

Is this "polynomial"?

Suppose capacities $c \in [1, U]^E$
(upper bound)

$F^* \leq mU \Rightarrow$ runtime $= O(m^2 U)$

But, input length $= O(m \log(U))$ ...
"pseudopolynomial"

$\boxed{\text{Idea 2: Widest path}}$

Claim: if maxflow value $= F^*$

Then widest path has $w \geq \frac{F^*}{m}$.

Proof sketch:

Key idea is flow decomposition into:

- "paths"
- "circulations" $(\partial f(v) = 0 \;\forall v)$

Aside: flow decomposition

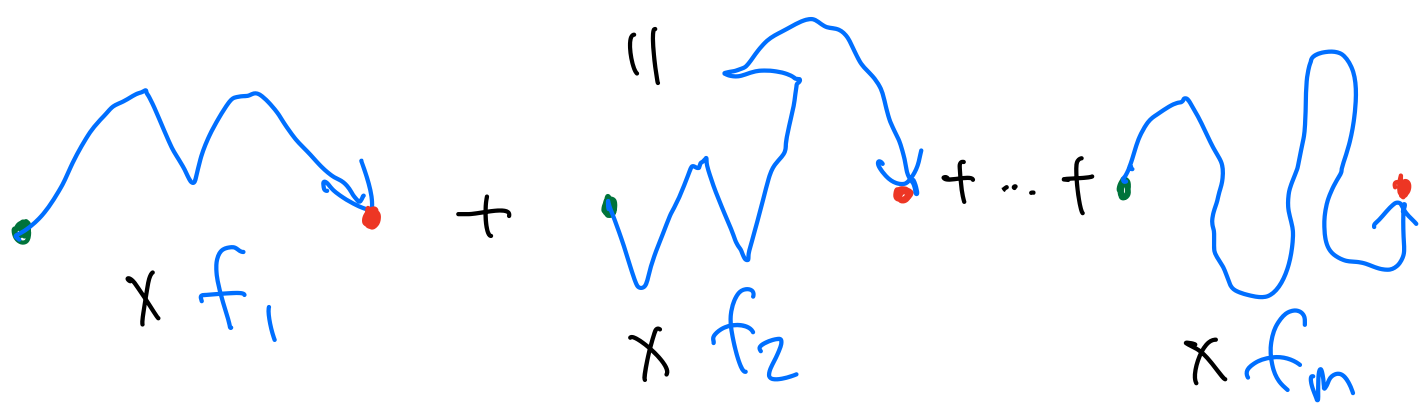
Every s-t flow can be decomposed into:

- $\leq m$ s-t paths $\times f_i$ ← flow value, path i
- one circulation $\left(\partial f(v) = 0 \text{ for all } v\right)$

If $\partial f(s)$ is integer, then so are all f_i.

any s-t flow

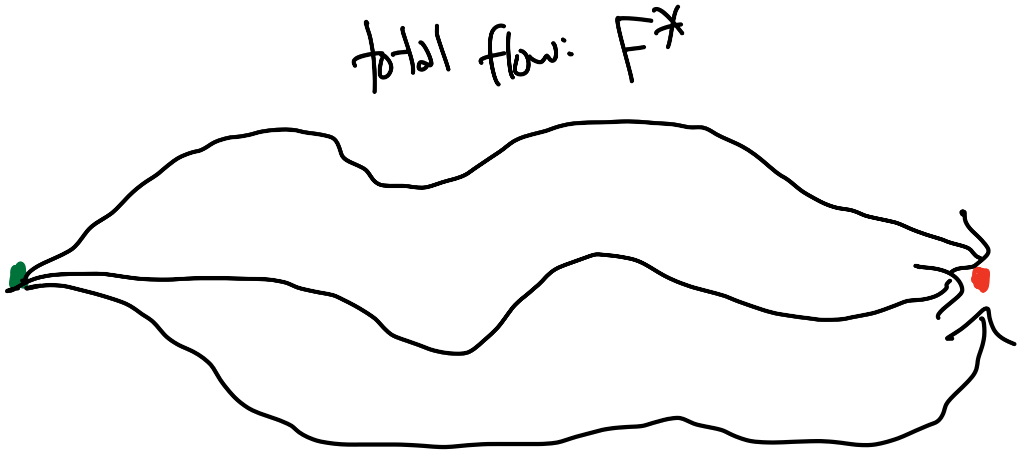
value $= \partial f(s)$



Very useful for flow reductions.

Thus, some path can carry $\frac{F^*}{m}$ flow

$\Rightarrow$ max width $\geq \frac{F^*}{m}$ as claimed.



Every iter decreases G^f maxflow $\left(1-\frac{1}{m}\right)\times$

after $m\log(F^*)$ iters,

$$\text{maxflow} \leq \left(1-\frac{1}{m}\right)^{m\log(F^*)} F^*$$

$$\leq \exp(-\log(F^*))\, F^* \leq 1$$

(any path works)

Runtime: $O(m^2 \log(F^*))$ "weakly polynomial"

The frontier

Best strongly polynomial: $O(mn)$

(King-Rao-Tarjan '94, Orlin '13)

Best weakly polynomial: $O(m^{1+o(1)} \log(U))$

[Chen-Kyng-Liu-Peng-Probst Gutenberg-Sachdeva '22]

Best pseudopolynomial: $\tilde{O}\left(m + \sqrt{mn}(F^*)^{3/4}\right)$

(Sidford-Tian '18)

($\tilde{O}$ = hide polylog(n))